Efficient Weight Computation

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1 Weights of Light Transport Paths

Given a light subpath $\bar{x}_i$ and a camera subpath $\bar{x}_o$ with n_i and n_o vertices respectively, our bidirectional estimator combines $2n_in_o$ estimators of the form $f(\bar{y}_{s,t}^{(u)})/p_{s,t}^{(u)}(\bar{y}_{s,t}^{(u)})$ with $s \in \{1, 2, \ldots, n_i\}$, $t \in \{1, 2, \ldots, n_o\}$, and $u \in \{0, 1\}$ via the multiple importance sampling (MIS) framework. This yields a combined estimator:

$$\sum_{s=1}^{n_i} \sum_{t=1}^{n_o} \sum_{u=0}^{1} w_{s,t}^{(u)}(\bar{y}_{s,t}^{(u)}) \frac{f(\bar{y}_{s,t}^{(u)})}{p_{s,t}^{(u)}(\bar{y}_{s,t}^{(u)})}, \tag{1}$$

where the weight $w_{s,t}^{(u)}$, when using the balanced heuristics [Veach 1997], is given by

$$w_{s,t}^{(u)}(\bar{y}_{s,t}) = \left(\sum_{s'=1}^{s+t-1} \sum_{u'=0}^{1} \frac{p_{s',s+t-s'}^{(u')}(\bar{y}_{s,t})}{p_{s,t}^{(u)}(\bar{y}_{s,t})} \right)^{-1} \tag{2}$$

for any path $\bar{y}_{s,t}$ with $(s + t)$ vertices.

Notice that, compared to standard bidirectional path tracing that combines n_in_o estimators, our position-free formulation offers twice the number of estimators since the direction connecting two depths is not unique (see Eq. (16) of the main paper).

2 Efficient Weight Computation

Computing Eqs. (1) and (2) for all s and t naïvely has a time complexity of $O(n_in_o(n_i + n_o))$ and is too slow to be practical. We now present our method that runs in $O(n_in_o)$ time. Our approach is conceptually similar to Veach's method for standard BDPT but differs in the exact mathematical form due to our position-free path formulation (see §4 for the paper).

Let $\bar{y}_{s,t} = (\boldsymbol{d}_0, z_1, \boldsymbol{d}_1, \ldots, z_n, \boldsymbol{d}_n)$ with $n = s + t$. For all $s', t' \in \{1, 2, \ldots, n\}$, define

$$p^{(0)}_{s'} := \prod_{i=1}^{s'-1} p(\boldsymbol{d}_i \mid z_i, \boldsymbol{d}_{i-1})\, p(z_{i+1} \mid z_i, \boldsymbol{d}_i), \tag{3}$$

$$p^{(1)}_{t'} := \prod_{i=n-t'+1}^{n-1} p(-\boldsymbol{d}_i \mid z_{i+1}, -\boldsymbol{d}_{i+1})\, p(z_i \mid z_{i+1}, -\boldsymbol{d}_i), \tag{4}$$

which denote the probability for constructing two subpaths containing the first s' and last t' vertices of $\bar{y}$, respectively. Then, for all u', s' and t', it holds that

$$p^{(u')}_{s',t'}(\bar{y}_{s,t}) = p^{(0)}_{s'}\, p^{(1)}_{t'}\, q^{(u')}_{s'}, \tag{5}$$

where

$$q^{(u')}_{s'} := \begin{cases} p(\boldsymbol{d}_{s'} \mid z_{s'}, \boldsymbol{d}_{s'-1}) & \text{if } u = 0, \\ p(-\boldsymbol{d}_{s'} \mid z_{s'+1}, -\boldsymbol{d}_{s'+1}) & \text{if } u = 1. \end{cases} \tag{6}$$

It follows that

$$\frac{p^{(u')}_{s',t'}(\bar{y}_{s,t})}{p^{(u)}_{s,t}(\bar{y}_{s,t})} = \frac{p^{(0)}_{s'}\, p^{(1)}_{t'}\, q^{(u')}_{s'}}{p^{(0)}_{s}\, p^{(1)}_{t}\, q^{(u)}_{s}}. \tag{7}$$

Note that, for any $s' < s$, we have

$$p^{(u')}_{s',t'}(\bar{y}_{s,t}) = p^{(0)}_{s'}\, \frac{p^{(1)}_{t'}}{p^{(1)}_{t+1}}\, q^{(u')}_{s'}\, p^{(1)}_{t+1}. \tag{8}$$

It follows that

$$\sum_{s'=1}^{s-1} \sum_{u'=0}^{1} \frac{p^{(u')}_{s',t'}(\bar{y}_{s,t})}{p^{(u)}_{s,t}(\bar{y}_{s,t})} = \frac{p^{(1)}_{t+1}}{p^{(1)}_{t}\, q^{(u)}_{s}} \underbrace{\sum_{s'=1}^{s-1} \sum_{u'=0}^{1} \frac{p^{(0)}_{s'}\, \frac{p^{(1)}_{t'}}{p^{(1)}_{t+1}} q^{(u')}_{s'}}{p^{(0)}_{s}}}_{=:\, P^{(0)}_s}. \tag{9}$$

Since

$$\frac{p^{(1)}_{t'}}{p^{(1)}_{t+1}} = \prod_{i=s'+1}^{s-1} p(-\boldsymbol{d}_i \mid z_{i+1}, -\boldsymbol{d}_{i+1})\, p(z_i \mid z_{i+1}, -\boldsymbol{d}_i), \tag{10}$$

it is easy to verify that $P^{(0)}_s$ depends only on depths $z_{s'}$ and directions $\boldsymbol{d}_{s'}$ with $s' \leq s$, which are all from the subpath $\bar{x}_i$. Further, $P^{(0)}_{s'}$ remains constant for all paths $\bar{y}_{s,t}$ with $s > s'$. This allows us to precompute $P^{(0)}_s$ using $\bar{x}_i$ for $s = 1, 2, \ldots, n_i$. To this end, $P^{(0)}_s(\bar{y})$ can be efficiently evaluated using the following relation:

$$P^{(0)}_s = \begin{cases} 0 & (s = 0), \\ \frac{p^{(0)}_{s-1}}{p^{(0)}_{s}} \left(P^{(0)}_{s-1} \frac{p^{(1)}_{t+2}}{p^{(1)}_{t+1}} + \sum_{u'} q^{(u')}_{s-1} \right) & (s > 1). \end{cases} \tag{11}$$

Using Eq. (11), we can compute $P^{(0)}_s(\bar{x}_i)$ for $s = 1, 2, \ldots, n_i$ in $O(n_i)$ time.

Similarly, for all $t' < t$, we have

$$\sum_{t'=1}^{t-1} \sum_{u'=0}^{1} \frac{p_{s',t'}^{(u')}(\bar{y}_{s,t})}{p_{s,t}^{(u)}(\bar{y}_{s,t})} = \frac{p_{s+1}^{(0)}}{p_s^{(0)} \, q_s^{(u)}} \underbrace{\sum_{t'=1}^{t-1} \sum_{u'=0}^{1} \frac{\frac{p_{s'}^{(0)}}{p_{s+1}^{(0)}} p_{t'}^{(1)} \, q_{n-t'}^{(u')}}{p_t^{(1)}}}_{=: \, P_t^{(1)}}, \tag{12}$$

where $P_t^{(1)}$ only depends on $\bar{x}_o$ can be computed in $O(n_o)$ time.

With both $P_s^{(0)}$ and $P_t^{(1)}$ precomputed, Eq. (2) becomes

$$w_{s,t}^{(u)}(\bar{y}_{s,t}) = \left(1 + P_s^{(0)} + P_t^{(1)} + \sum_{u'=0}^{1} \frac{p_{s-1,t+1}^{(u')}(\bar{y}_{s,t}) + p_{s+1,t-1}^{(u')}(\bar{y}_{s,t})}{p_{s,t}^{(u)}(\bar{y}_{s,t})} \right)^{-1}, \tag{13}$$

which can be computed in constant time. This leads to a full bidirectional estimator with time complexity $O(n_i n_o)$.