A Differential Monte Carlo Solver For the Poisson Equation: Supplemental Document

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1 DERIVATION OF THE INTERIOR TERM

In what follows, we derive Eq. (20) of the paper.

We first apply integration by parts to the left-hand side of this equation, obtaining

$$-\int_{\hat{\Omega}} \mathcal{G}^{\hat{\Omega}}\,\Delta_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\mathrm{d}\hat{\Omega} = -\int_{\hat{\Omega}} \nabla_{\hat{\Omega}}\cdot\left(\mathcal{G}^{\hat{\Omega}}\nabla_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\right)\mathrm{d}\hat{\Omega} + \int_{\hat{\Omega}}\left(\nabla_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}\right)\cdot\nabla_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\mathrm{d}\hat{\Omega}. \tag{1}$$

where $\mathcal{G}^{\hat{\Omega}}$ denotes $\mathcal{G}^{\hat{\Omega}}(\boldsymbol{p}\leftrightarrow\boldsymbol{q})$ as a function of $\boldsymbol{q}$ (i.e., with $\boldsymbol{p}$ fixed).

Applying the divergence theorem to the first integral on the right-hand side of Eq. (1) gives

$$\begin{aligned}&-\int_{\hat{\Omega}} \nabla_{\hat{\Omega}}\cdot\left(\mathcal{G}^{\hat{\Omega}}\,\nabla_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\right)\mathrm{d}\hat{\Omega}\\ &= -\int_{\partial\hat{\Omega}} \mathcal{G}^{\hat{\Omega}}\left(-\boldsymbol{n}\cdot\nabla_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\right)\mathrm{d}\partial\hat{\Omega} = 0,\end{aligned} \tag{2}$$

where the second equality follows the fact that $\mathcal{G}^{\hat{\Omega}}(\boldsymbol{p}\leftrightarrow\boldsymbol{s}) = 0$ for all $\boldsymbol{s}$ on the domain boundary $\partial\hat{\Omega}$.

For the second term on the right-hand side of Eq. (1), we also apply integration by parts followed by the divergence theorem, yielding

$$\begin{aligned}&\int_{\hat{\Omega}}\left(\nabla_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}\right)\cdot\nabla_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\mathrm{d}\hat{\Omega}\\ &= \int_{\hat{\Omega}}\nabla_{\hat{\Omega}}\cdot\left(\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\nabla_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}\right)\mathrm{d}\hat{\Omega} - \int_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\underbrace{\left(\nabla_{\hat{\Omega}}\cdot\nabla_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}\right)}_{=\Delta_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}}\mathrm{d}\hat{\Omega}\\ &= \int_{\partial\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\underbrace{\left(-\boldsymbol{n}\cdot\nabla_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}\right)}_{=-\mathcal{P}^{\hat{\Omega}}}\mathrm{d}\partial\hat{\Omega} - \int_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right)\left(-\delta_{\boldsymbol{p}}\right)\mathrm{d}\hat{\Omega}\\ &= -\int_{\partial\hat{\Omega}}\mathcal{P}^{\hat{\Omega}}(\boldsymbol{p}\to\boldsymbol{s})\left(\boldsymbol{v}(\boldsymbol{s})\cdot(\nabla_{\hat{\Omega}}\hat{u})(\boldsymbol{s})\right)\mathrm{d}\boldsymbol{s} + \boldsymbol{v}(\boldsymbol{p})\cdot\nabla_{\hat{\Omega}}\hat{u}(\boldsymbol{p}),\end{aligned} \tag{3}$$

where $\boldsymbol{n}$ is the inward unit-normal.

Lastly, we obtain Eq. (20) of the paper by adding Eqs. (2) and (3).

Relaxing assumptions. As stated in the paper, we assume that the source function f is independent of the parameter θ to simplify the derivation. With this assumption relaxed, Eq. (17) of the paper becomes

$$\Delta_{\hat{\Omega}}(\partial_\theta\hat{u}) = \Delta_{\hat{\Omega}}\left(\boldsymbol{v}\cdot\nabla_{\hat{\Omega}}\hat{u}\right) - \partial_\theta f, \tag{4}$$

which in turn causes Eq. (22) of the paper to also include an *interior* component:

$$(\partial_\theta u)(\boldsymbol{p}) = \underbrace{-\int_{\hat{\Omega}}\mathcal{G}^{\hat{\Omega}}(\boldsymbol{p}\leftrightarrow\boldsymbol{q})\,[(\partial_\theta f)(\boldsymbol{q})]\,\mathrm{d}\boldsymbol{q}}_{\text{interior}} + \underbrace{\int_{\partial\hat{\Omega}}\mathcal{P}^{\hat{\Omega}}(\boldsymbol{p}\to\boldsymbol{s})\left(\partial_\theta\,\hat{g}(\boldsymbol{s}) - \boldsymbol{v}(\boldsymbol{s})\cdot\nabla_{\hat{\Omega}}\hat{u}(\boldsymbol{s})\right)\mathrm{d}\boldsymbol{s}}_{\text{boundary}}. \tag{5}$$

2 DIFFERENTIAL KERNELS AND CONTROL VARIATES

Let $\boldsymbol{s}_l \coloneqq \boldsymbol{s} + l\,\boldsymbol{n}(\boldsymbol{s})$ be a point for all $l > 0$. Then, $\boldsymbol{s}_l$ resides in the interior of the domain $\hat{\Omega}$ when l is sufficiently small. In this case, it holds that

$$u(\boldsymbol{s}_l) = \underbrace{\int_{B_{\boldsymbol{c}}} f(\boldsymbol{y})\,G^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\leftrightarrow\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}_{\text{interior}} + \underbrace{\int_{\partial B_{\boldsymbol{c}}} u(\boldsymbol{z})\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}}_{\text{boundary}}, \tag{6}$$

where $G^{B_{\boldsymbol{c}}}$ and $P^{B_{\boldsymbol{c}}}$ are Green's function and Poisson kernel associated with the ball $B_{\boldsymbol{c}}$.

2.1 Differential Kernels

Differentiating both sides of Eq. (6) with respect to l and evaluating at $l = l_0$ produces:

$$\left[\frac{\mathrm{d}}{\mathrm{d}l}u(\boldsymbol{s}_l)\right]_{l=l_0} = \underbrace{\int_{B_{\boldsymbol{c}}} f(\boldsymbol{y})\left[\frac{\mathrm{d}}{\mathrm{d}l}G^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\leftrightarrow\boldsymbol{y})\right]_{l=l_0}\mathrm{d}\boldsymbol{y}}_{\text{interior}} + \underbrace{\int_{\partial B_{\boldsymbol{c}}} u(\boldsymbol{z})\left[\frac{\mathrm{d}}{\mathrm{d}l}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{z})\right]_{l=l_0}\mathrm{d}\boldsymbol{z}}_{\text{boundary}}, \tag{7}$$

where the left-hand side gives the directional derivative of u at $\boldsymbol{s}_{l_0}$ in the direction $\boldsymbol{n}(\boldsymbol{s})$. In other words, $\left[\frac{\mathrm{d}}{\mathrm{d}l}u(\boldsymbol{s}_l)\right]_{l=l_0} = \boldsymbol{n}(\boldsymbol{s})\cdot\nabla u(\boldsymbol{s}_{l_0})$.

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It follows that

$$\left[\frac{\mathrm{d}}{\mathrm{d}l}u(\boldsymbol{s}_l)\right]_{l=0} = \boldsymbol{n}(\boldsymbol{s})\cdot\nabla u(\boldsymbol{s}) = \partial_{\boldsymbol{n}(\boldsymbol{s})}u. \tag{8}$$

Further, according to Eq. (4) of the paper, it holds that

$$\left[\frac{\mathrm{d}}{\mathrm{d}l}G^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l \leftrightarrow \boldsymbol{y})\right]_{l=0} = P^{B_{\boldsymbol{c}}}(\boldsymbol{y}\to\boldsymbol{s}). \tag{9}$$

Lastly, taking the limits of $l_0 \downarrow 0$ on both sides of Eq. (7) produces Eq. (25) of the paper with the differential kernel $\partial_{\boldsymbol{n}(\boldsymbol{s})}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}\to\boldsymbol{z})$ defined as

$$\partial_{\boldsymbol{n}(\boldsymbol{s})}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}\to\boldsymbol{z}) := \lim_{l_0\downarrow 0}\left[\frac{\mathrm{d}}{\mathrm{d}l}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l \leftrightarrow \boldsymbol{z})\right]_{l=l_0}. \tag{10}$$

In practice, we evaluate the right-hand side of Eq. (10) by symbolically differentiating the Poisson kernel $P^{B_{\boldsymbol{c}}}$ and evaluating the result at $l = 0$.

2.2 Our Control Variates

We now derive the control variates in Eq. (26) of the main paper from Eq. (25).

Interior component. Let h be the solution of the following Poisson problem over the ball $B_{\boldsymbol{c}}$ with constant source and boundary functions:

$$\begin{aligned} \Delta h &= -1 \quad \text{on } B_{\boldsymbol{c}}, \\ h &= 0 \quad \text{on } \partial B_{\boldsymbol{c}}. \end{aligned} \tag{11}$$

Then, according to the representation formula, we have

$$h(\boldsymbol{s}_l) = \underbrace{\int_{B_{\boldsymbol{c}}} G^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l \leftrightarrow \boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}_{\text{interior}}, \tag{12}$$

for any positive l near zero. Differentiating both sides of this equation with respect to l and evaluating at $l = 0$ produces:

$$\left[\frac{\mathrm{d}}{\mathrm{d}l}h(\boldsymbol{s}_l)\right]_{l=0} = \int_{B_{\boldsymbol{c}}} P^{B_{\boldsymbol{c}}}(\boldsymbol{y}\to\boldsymbol{s})\,\mathrm{d}\boldsymbol{y}. \tag{13}$$

On the other hand, it is easy to verify that the Poisson equation (11) has the analytical solution

$$h(\boldsymbol{x}) = \frac{R^2 - \|\boldsymbol{x}-\boldsymbol{c}\|^2}{2n}, \tag{14}$$

where R and n denote the radius and dimensionality of the ball $B_{\boldsymbol{c}}$, respectively.

Evaluating the left-hand side of Eq. (13) using Eq. (14) yields

$$\int_{B_{\boldsymbol{c}}} P^{B_{\boldsymbol{c}}}(\boldsymbol{y}\to\boldsymbol{s})\,\mathrm{d}\boldsymbol{y} = \frac{R}{n}, \tag{15}$$

which in turn produces the *interior* component of Eq. (27) of the main paper via

$$\begin{aligned} &\int_{B_{\boldsymbol{c}}} f(\boldsymbol{y})\,P^{B_{\boldsymbol{c}}}(\boldsymbol{y}\to\boldsymbol{s})\,\mathrm{d}\boldsymbol{y} \\ =& \int_{B_{\boldsymbol{c}}} (f(\boldsymbol{y}) - f(\boldsymbol{s}) + f(\boldsymbol{s}))\,P^{B_{\boldsymbol{c}}}(\boldsymbol{y}\to\boldsymbol{s})\,\mathrm{d}\boldsymbol{y} \\ =& \int_{B_{\boldsymbol{c}}} (f(\boldsymbol{y}) - f(\boldsymbol{s}))\,P^{B_{\boldsymbol{c}}}(\boldsymbol{y}\to\boldsymbol{s})\,\mathrm{d}\boldsymbol{y} + f(\boldsymbol{s})\,\frac{R}{n}. \end{aligned} \tag{16}$$

Boundary component. Since the Poisson kernel $P^{B_{\boldsymbol{c}}}(\boldsymbol{s}\to\boldsymbol{z})$ is essentially a probability distribution of $\boldsymbol{z}$ over the boundary $\partial B_{\boldsymbol{c}}$ (with $\boldsymbol{s}$ fixed), it holds that

$$\begin{aligned} &\int_{\partial B_{\boldsymbol{c}}} \left[\frac{\mathrm{d}}{\mathrm{d}l}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{z})\right]_{l=l_0}\mathrm{d}\boldsymbol{z} \\ =& \left[\frac{\mathrm{d}}{\mathrm{d}l}\underbrace{\int_{\partial B_{\boldsymbol{c}}} P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}}_{\equiv 1}\right]_{l=l_0} \equiv 0, \end{aligned} \tag{17}$$

for any l_0. Thus, we have

$$\int_{\partial B_{\boldsymbol{c}}} \partial_{\boldsymbol{n}(\boldsymbol{s})}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}\to\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} \equiv 0, \tag{18}$$

which allows us to subtract $g(\boldsymbol{s})\int_{\partial B_{\boldsymbol{c}}} \partial_{\boldsymbol{n}(\boldsymbol{s})}P^{B_{\boldsymbol{c}}}(\boldsymbol{s}\to\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}$ from the *boundary* component of Eq. (25) of the paper, producing that of Eq. (26).

2.3 Proof of Convergence

In the following, we show that the integrals of Eq. (26) resulting from our control variates converge.

Interior integral. Without loss of generality, we examine the integrand of Eq. (26)'s *interior* component when l approaches zero:

$$\lim_{l\downarrow 0}\Big((f(\boldsymbol{s}_l) - f(\boldsymbol{s}))\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\Big). \tag{19}$$

Applying Taylor expansion to $f(\boldsymbol{s}_l)$—which we consider as a function of l—at $l = 0$ produces

$$f(\boldsymbol{s}_l) = f(\boldsymbol{s}) + l\,\partial_{\boldsymbol{n}(\boldsymbol{s})}f(\boldsymbol{s}) + o(l^2). \tag{20}$$

Thus, we have

$$\begin{aligned} &\lim_{l\downarrow 0}\Big((f(\boldsymbol{s}_l) - f(\boldsymbol{s}))\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\Big) \\ =& \lim_{l\downarrow 0}\Big(\Big(l\,\partial_{\boldsymbol{n}(\boldsymbol{s})}f(\boldsymbol{s}) + o(l^2)\Big)\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\Big) \\ =& \partial_{\boldsymbol{n}(\boldsymbol{s})}f(\boldsymbol{s})\lim_{l\downarrow 0}\Big(l\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\Big) + \lim_{l\downarrow 0}\Big(o(l^2)\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\Big). \end{aligned} \tag{21}$$

When $B_{\boldsymbol{c}}$ is a 2D or 3D sphere, the second-order term in this equation converges:

$$\lim_{l\downarrow 0}\Big(o(l^2)\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\Big) < \infty. \tag{22}$$

The first-order term $\lim_{l\downarrow 0}\left(l\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})\right)$, on the other hand, behaves differently between 2D and 3D. In 2D, the limit equals a finite value; In 3D, $l\,P^{B_{\boldsymbol{c}}}(\boldsymbol{s}_l\to\boldsymbol{s})$ is a weak singularity as l approaches zero, but integrating this term over the interior of the sphere $B_{\boldsymbol{c}}$ produces a finite result.

Boundary integral. We now show the convergence of the *boundary* integral of Eq. (26) of the paper. We focus on the 2D case since the 3D case follows a similar proof.

We first parameterize the boundary $\partial B_{\boldsymbol{c}}$ using $\boldsymbol{z}_\phi$ such that the angle between the vectors $(\boldsymbol{s}-\boldsymbol{c})$ and $(\boldsymbol{z}_\phi-\boldsymbol{c})$ equals ϕ. Under this parameterization, it holds that $\boldsymbol{z}_0 = \boldsymbol{s}$.

For any ϕ, applying Taylor expansion at $\phi = 0$ gives

$$u(z_\phi) = g(s) + \phi\, \partial_{t(s)} u(s) + o(\phi^2), \tag{23}$$

$$u(z_{-\phi}) = g(s) - \phi\, \partial_{t(s)} u(s) + o(\phi^2), \tag{24}$$

where $\partial_{t(s)} u(s)$ denotes the tangential derivative of u at s. It follows that

$$\begin{aligned}
&\int_{\partial B_c} (u(z) - g(s))\, \partial_{n(s)} P^{B_c}(s \to z)\, \mathrm{d}z \\
&= R \int_{-\pi}^{\pi} (u(z_\phi) - g(s))\, \partial_{n(s)} P^{B_c}(s \to z_\phi)\, \mathrm{d}\phi \\
&= R \int_{0}^{\pi} (u(z_\phi) + u(z_{-\phi}) - 2g(s))\, \partial_{n(s)} P^{B_c}(s \to z_\phi)\, \mathrm{d}\phi \\
&= R \int_{0}^{\pi} o(\phi^2)\, \partial_{n(s)} P^{B_c}(s \to z_\phi)\, \mathrm{d}\phi.
\end{aligned} \tag{25}$$

When B_c is a 2D ball, the integrand $o(\phi^2)\, \partial_{n(s)} P^{B_c}(s \to z_\phi)$ converges when ϕ approaches zero:

$$\lim_{\phi \downarrow 0} o(\phi^2)\, \partial_{n(s)} P^{B_c}(s \to z_\phi) = \lim_{\phi \downarrow 0} \frac{o(\phi^2)}{2\pi(\cos\phi - 1)} < \infty, \tag{26}$$

which ensures the convergence of the *boundary* integral.