1 DERIVATION OF THE INTERIOR TERM

In what follows, we derive Eq. (20) of the paper.

We first apply integration by parts to the left-hand side of this equation, obtaining

\[
- \int_\Omega G^\Omega \Delta \hat{u} \, d\hat{\Omega} = - \int_\Omega \nabla \cdot \left( G^\Omega \nabla \left( \sigma \cdot \nabla \hat{u} \right) \right) d\Omega + \int_{\partial \Omega} \nabla G^\Omega \cdot \nabla \left( \sigma \cdot \nabla \hat{u} \right) d\hat{\Omega},
\]

where \( G^\Omega \) denotes \( G^\Omega (p \leftrightarrow q) \) as a function of \( q \) (i.e., with \( p \) fixed).

Applying the divergence theorem to the first integral on the right-hand side of Eq. (1) gives

\[
- \int_\Omega \nabla \cdot \left( G^\Omega \nabla \left( \sigma \cdot \nabla \hat{u} \right) \right) d\Omega = - \int_{\partial \Omega} G^\Omega \left( -\sigma \cdot \nabla \hat{u} \right) d\hat{\Omega} = 0,
\]

where the second equality follows the fact that \( G^\Omega (p \leftrightarrow s) = 0 \) for all \( s \) on the domain boundary \( \partial \Omega \).

For the second term on the right-hand side of Eq. (1), we also apply integration by parts followed by the divergence theorem, yielding

\[
\int_{\partial \Omega} \nabla G^\Omega \cdot \nabla \left( \sigma \cdot \nabla \hat{u} \right) d\hat{\Omega} = \int_\Omega \nabla \cdot \left( \left( \sigma \cdot \nabla \hat{u} \right) \nabla G^\Omega \right) d\Omega - \int_\Omega \nabla \cdot \left( \nabla \hat{u} \nabla G^\Omega \right) d\Omega
\]

\[
= \int_{\partial \Omega} \left( \sigma \cdot \nabla \hat{u} \right) \left( -n \cdot \nabla G^\Omega \right) d\hat{\Omega} - \int_\Omega \left( \sigma \cdot \nabla \hat{u} \right) \left( -\delta_p \right) d\Omega
\]

\[
= - \int_{\partial \Omega} P^\Omega (p \rightarrow s) \left( \sigma (s) \cdot \nabla \hat{u} (s) \right) ds + \sigma (p) \cdot \nabla \hat{u} (p),
\]

where \( n \) is the inward unit-normal.

Lastly, we obtain Eq. (20) of the paper by adding Eqs. (2) and (3).

Relaxing assumptions. As stated in the paper, we assume that the source function \( f \) is independent of the parameter \( \theta \) to simplify the derivation. With this assumption relaxed, Eq. (17) of the paper becomes

\[
\Delta \Omega \left( \partial_\theta \hat{u} \right) = \Delta \Omega \left( \sigma \cdot \nabla \hat{u} \right) - \partial_\theta f,
\]

which in turn causes Eq. (22) of the paper to also include an interior component:

\[
(\partial_\theta u)(p) = - \int_\Omega G^\Omega (p \leftrightarrow q) \left[ (\partial_\theta f)(q) \right] dq + \int_{\partial \Omega} F^\Omega (p \rightarrow s) \left( \partial_\theta \hat{g}(s) - \sigma (s) \cdot \nabla \hat{u} (s) \right) ds.
\]

2 DIFFERENTIAL KERNELS AND CONTROL VARIATES

Let \( s_l := s + ln(s) \) be a point for all \( l > 0 \). Then, \( s_l \) resides in the interior of the domain \( \Omega \) when \( l \) is sufficiently small. In this case, it holds that

\[
\begin{align*}
\left. u(s_l) \right|_{l=l_0} &= \int_{B_{l_0}} f(y) G^{B_{l_0}} (s_l \leftrightarrow y) \, dy + \\
\left. \int_{\partial B_{l_0}} u(z) p^{B_{l_0}} (s_l \rightarrow z) \, dz \right|_{l=l_0},
\end{align*}
\]

where \( G^{B_{l_0}} \) and \( p^{B_{l_0}} \) are Green’s function and Poisson kernel associated with the ball \( B_{l_0} \).

2.1 Differential Kernels

Differentiating both sides of Eq. (6) with respect to \( l \) and evaluating at \( l = l_0 \) produces:

\[
\left. \frac{d}{dl} u(s_l) \right|_{l=l_0} = \int_{B_{l_0}} f(y) \left. \frac{d}{dl} G^{B_{l_0}} (s_l \leftrightarrow y) \right|_{l=l_0} \, dy + \\
\left. \int_{\partial B_{l_0}} u(z) \frac{d}{dl} p^{B_{l_0}} (s_l \rightarrow z) \right|_{l=l_0} \, dz,
\]

where the left-hand side gives the directional derivative of \( u \) at \( s_{l_0} \) in the direction \( n(s) \). In other words, \( \left. \frac{d}{dl} u(s_l) \right|_{l=l_0} = n(s) \cdot \nabla u(s_{l_0}). \)
It follows that
\[
\left. \frac{d}{dt} u(s) \right|_{t=0} = n(s) \cdot \nabla u(s) = \partial_n(s) u. \tag{8}
\]

Further, according to Eq. (4) of the paper, it holds that
\[
\left. \frac{d}{dt} G^{Bc}(s_1 \leftrightarrow y) \right|_{t=0} = p^{Bc}(y \rightarrow s). \tag{9}
\]

Lastly, taking the limits of \( l \downarrow 0 \) on both sides of Eq. (7) produces Eq. (25) of the paper with the differential kernel \( \partial_n(s) p^{Bc}(s \rightarrow z) \) defined as
\[
\partial_n(s) p^{Bc}(s \rightarrow z) := \lim_{l \downarrow 0} \left. \frac{d}{dt} p^{Bc}(s_l \leftrightarrow z) \right|_{t=0}. \tag{10}
\]

In practice, we evaluate the right-hand side of Eq. (10) by symbolically differentiating the Poisson kernel \( p^{Bc} \) and evaluating the result at \( l = 0 \).

### 2.2 Our Control Variates

We now derive the control variates in Eq. (26) of the main paper from Eq. (25).

**Interior component.** Let \( h \) be the solution of the following Poisson problem over the ball \( B_c \) with constant source and boundary functions:
\[
\begin{align*}
\Delta h &= -1 & \text{on } B_c, \\
h &= 0 & \text{on } \partial B_c.
\end{align*}
\tag{11}
\]

Then, according to the representation formula, we have
\[
h(s) = \int_{Bc} G^{Bc}(s \leftrightarrow y) \, dy. \tag{12}
\]

for any positive \( l \) near zero. Differentiating both sides of this equation with respect to \( l \) and evaluating at \( l = 0 \) produces:
\[
\left. \frac{d}{dl} h(s_1) \right|_{l=0} = \int_{Bc} p^{Bc}(y \rightarrow s) \, dy. \tag{13}
\]

On the other hand, it is easy to verify that the Poisson equation (11) has the analytical solution
\[
h(x) = \frac{R^2 - \|x - c\|^2}{2n}. \tag{14}
\]

where \( R \) and \( n \) denote the radius and dimensionality of the ball \( B_c \), respectively.

Evaluating the left-hand side of Eq. (13) using Eq. (14) yields
\[
\int_{Bc} p^{Bc}(y \rightarrow s) \, dy = \frac{R}{n}, \tag{15}
\]

which in turn produces the **interior** component of Eq. (27) of the main paper via
\[
\int_{Bc} f(y) p^{Bc}(y \rightarrow s) \, dy = \int_{Bc} (f(y) - f(s) + f(s)) p^{Bc}(y \rightarrow s) \, dy \tag{16}
\]

\[
= \int_{Bc} (f(y) - f(s)) p^{Bc}(y \rightarrow s) \, dy + f(s) \frac{R}{n},
\]

**Boundary component.** Since the Poisson kernel \( p^{Bc}(s \rightarrow z) \) is essentially a probability distribution of \( z \) over the boundary \( \partial B_c \) (with \( s \) fixed), it holds that
\[
\int_{\partial B_c} \frac{d}{dl} p^{Bc}(s_l \rightarrow z) \, dz \left|_{l=0} \equiv 1 \right.
\]

for any \( l_0 \). Thus, we have
\[
\int_{\partial B_c} \partial_n(s) p^{Bc}(s \rightarrow z) \, dz \equiv 0, \tag{17}
\]

which allows us to subtract \( g(s) \int_{\partial B_c} \partial_n(s) p^{Bc}(s \rightarrow z) \, dz \) from the boundary component of Eq. (25) of the paper, producing that of Eq. (26).

### 2.3 Proof of Convergence

In the following, we show that the integrals of Eq. (26) resulting from our control variates converge.

**Interior integral.** Without loss of generality, we examine the integrand of Eq. (26)’s interior component when \( l \) approaches zero:
\[
\lim_{l \downarrow 0} \left( (f(s_l) - f(s)) p^{Bc}(s_l \rightarrow s) \right) . \tag{19}
\]

Applying Taylor expansion to \( f(s_l) \)—which we consider as a function of \( l \) at \( l = 0 \) produces
\[
f(s_l) = f(s) + l \partial_n(s) f(s) + o(l^2). \tag{20}
\]

Thus, we have
\[
\lim_{l \downarrow 0} \left( (f(s_l) - f(s)) p^{Bc}(s_l \rightarrow s) \right) = \lim_{l \downarrow 0} \left( l \partial_n(s) f(s) + o(l^2) \right) p^{Bc}(s_l \rightarrow s) \tag{21}
\]

When \( B_c \) is a 2D or 3D sphere, the second-order term in this equation converges:
\[
\lim_{l \downarrow 0} \left( o(l^2) p^{Bc}(s_l \rightarrow s) \right) < \infty. \tag{22}
\]

The first-order term \( \lim_{l \downarrow 0} (l p^{Bc}(s_l \rightarrow s)) \), on the other hand, behaves differently between 2D and 3D. In 2D, the limit equals a finite value; In 3D, \( l p^{Bc}(s_l \rightarrow s) \) is a weak singularity as \( l \) approaches zero, but integrating this term over the interior of the sphere \( B_c \) produces a finite result.

**Boundary integral.** We now show the convergence of the boundary integral of Eq. (26) of the paper. We focus on the 2D case since the 3D case follows a similar proof.

We first parameterize the boundary \( \partial B_c \) using \( z_\phi \) such that the angle between the vectors \( (s - c) \) and \( (z_\phi - c) \) equals \( \phi \). Under this parameterization, it holds that \( z_0 = s \).
For any $\phi$, applying Taylor expansion at $\phi = 0$ gives

\begin{align}
  u(z_\phi) &= g(s) + \phi \partial_{t(s)} u(s) + o(\phi^2), \\
  u(z_{-\phi}) &= g(s) - \phi \partial_{t(s)} u(s) + o(\phi^2),
\end{align}

(23)

(24)

where $\partial_{t(s)} u(s)$ denotes the tangential derivative of $u$ at $s$. It follows that

\[
\int_{\partial B_c} (u(z) - g(s)) \partial_{n(s)} P_{B_c}(s \to z) \, dz \\
= R \int_{-\pi}^{\pi} (u(z_\phi) - g(s)) \partial_{n(s)} P_{B_c}(s \to z_\phi) \, d\phi \\
= R \int_{0}^{\pi} (u(z_\phi) + u(z_{-\phi}) - 2g(s)) \partial_{n(s)} P_{B_c}(s \to z_\phi) \, d\phi \\
= R \int_{0}^{\pi} o(\phi^2) \partial_{n(s)} P_{B_c}(s \to z_\phi) \, d\phi.
\]

(25)

When $B_c$ is a 2D ball, the integrand $o(\phi^2) \partial_{n(s)} P_{B_c}(s \to z_\phi)$ converges when $\phi$ approaches zero:

\[
\lim_{\phi \to 0} o(\phi^2) \partial_{n(s)} P_{B_c}(s \to z_\phi) = \lim_{\phi \to 0} o(\phi^2) \frac{o(\phi^2)}{2\pi (\cos \phi - 1)} < \infty,
\]

(26)

which ensures the convergence of the boundary integral.